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A Naturally Ordered Enumeration of Compositions

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A composition of the positive integer M, with N-parts, is a vector of N non-negative integer components the sum of which is M. The paper presents a transformation that "enumerates" (assigns serial numbers) the set of all possible compositions most densely so that their "natural" ordering is preserved.

The transformation is useful for the efficient retrieval and compact storage of probabilities, or functions on probability spaces, in finite memories.

STATEMENT OF THE PROBLEM

Let $(l_N, l_{N-1}, \ldots, l_2, l_1)$ denote an N-dimensional vector whose ordered components $l_i \geqslant 0$ $i = 1, 2, \ldots, N$ are non-negative integers which satisfy the condition

$$\sum_{i=1}^{N} l_i = M. \tag{1}$$

It is well known (see, e.g., Feller [3] for "Bose Einstein Statistics" or Riordan [1]) that there exist $\binom{M+N-1}{M}$ distinct such vectors. Denote by $S(M, N)$ the set of all such vectors and define the lexicographic ordering relation on $S(M, N)$ as follows:
Let $L^1 = (l_N^1, l_{N-1}^1, \ldots, l_2^1, l_1^1)$ and $L^2 = (l_N^2, l_{N-1}^2, \ldots, l_2^2, l_1^2)$ be any pair of distinct vectors in $S(M, N)$. Then, we say that, $L^2 \succ L^1$ if and only if $l_j^2 > l_j^1$ where

$$j = \max_{i=1,\ldots,N} \{i: l_i^1 \neq l_i^2\}.$$

Next, denote by

$$I(M, N) = \left\{0, 1, 2, \ldots, \binom{M+N-1}{M} - 1\right\}$$

the set of the first $\binom{M+N-1}{M}$ non-negative integers.

Let it be required to find a transformation T: $S(M, N) \to I(M, N)$, that "enumerates" (or assigns serial numbers to) vectors in $S(M, N)$, and has the properties:

(a) T is one to one and onto.
(b) T preserves the natural lexicographic ordering

$$\prec (\text{i.e., if } L^2 \succ L^1;\ L^1, L^2 \in S(M, N), \qquad \text{then} \qquad T(L^2) > T(L^1)).$$

The Solution

We start by defining a transformation T which is later shown to have the required properties.

For any vector $L = (l_N, l_{N-1}, ..., l_2, l_1)$ and any integer i, $1 \leqslant i \leqslant M$, we define

$$n_i = \min_{n=1,2,...N} \left\{ n: \sum_{r=1}^{n} l_r \geqslant i \right\}.$$

As an interpretation of n_i we may think of the vector L as analogous to an array of N compartments in which M marbles are distributed. Suppose that the marbles are sequentially removed from compartments so that at each stage the one being removed is taken from the rightmost (lowest in order) compartment in which it could be found. Then n_i represents the compartment from which the i-th marble was removed.

Define also a weight function $g(\alpha, \beta)$; $0 \leqslant \alpha \leqslant M$; $1 \leqslant \beta$ by:

$$\begin{aligned} g(\alpha, \beta) &= \binom{\alpha + \beta - 2}{\alpha}, \qquad 1 \leqslant \alpha \leqslant M, \\ g(0, \beta) &= 1. \end{aligned} \tag{2}$$

We have the known identity:

$$g(0, \beta) + g(1, \beta) + ,..., g(\alpha, \beta) = g(\alpha, \beta + 1). \tag{3}$$

Then, the transformation T is defined by

$$T(L) = \sum_{i=1}^{M} g(i, n_i) = \sum_{i=l_1+1}^{M} \binom{i + n_i - 2}{i}. \tag{4}$$

The preservation of the ordering and the density of enumeration, for a complete set of compositions $S(4, 3)$, are illustrated by a table and a three-dimensional description in Fig. 1.

L	$T(L)$
(0, 0, 4)	0
(0, 1, 3)	1
(0, 2, 2)	2
(0, 3, 1)	3
(0, 4, 0)	4
(1, 0, 3)	5
(1, 1, 2)	6
(1, 2, 1)	7
(1, 3, 0)	8
(2, 0, 2)	9
(2, 1, 1)	10
(2, 2, 0)	11
(3, 0, 1)	12
(3, 1, 0)	13
(4, 0, 0)	14

$T: S(4, 3) \to I(4, 3)$

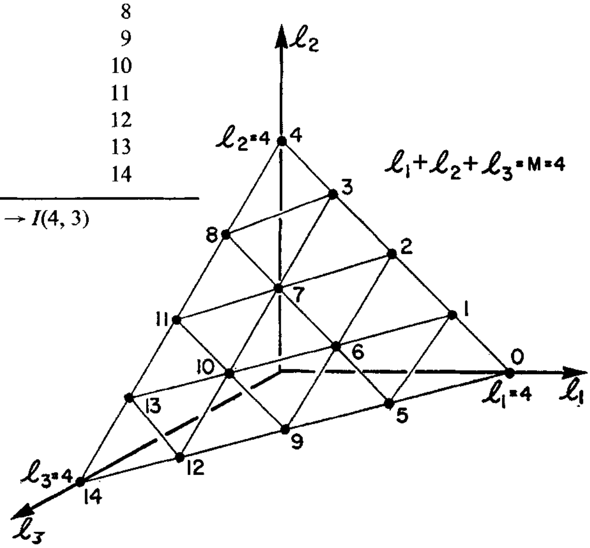

FIG. 1. The case $M = 4, N = 3$.

Note. A transformation τ which satisfies property (a) but does not obey the ordering condition (b) was given in Lehmer [2]. The transformation there, which is in closed form, uses a "combinatorial" expansion in which the vector $(l_N, l_{N-1}, ..., l_1)$ is transformed first into the form $(q_N, q_{N-1}, ..., q_1)$, where $q_i = i - 1 + \sum_{j=1}^{i} l_j$, and then

$$\tau(L) = \sum_{i=1}^{N-1} \binom{q_i}{i}.$$

LEMMA 1. *The transformation T, in* (4), *preserves the ordering in the sense that $T(L^2) > T(L^1)$ if $L^2 \succ L^1$.*

Proof. Pick any pair of vectors L^1, L^2 for which $L^2 \succ L^1$. Then there exist some

$$j = \max_{i=1,2,...} \{i: l_i^1 \neq l_i^2\}$$

where $l_j^2 > l_j^1$. Since

$$\sum_{i=1}^{N} l_i^1 = \sum_{i=1}^{N} l_i^2 = M,$$

we must have just $j \geqslant 2$ and that $\sum_{i=1}^{j} l_i^1 = \sum_{i=1}^{j} l_i^2$ so we denote

$$\overline{M} = \sum_{i=1}^{j-1} l_i^1, \qquad \text{where} \quad M > \overline{M} \geqslant 1.$$

By the definitions of T and$\overline{M}$ we may now compute

$$\begin{aligned}
T(L^2) - T(L^1) &= \sum_{i=1}^{M} g(i, n_i^2) - \sum_{i=1}^{M} g(i, n_i^1) \\
&= \sum_{i=1}^{\overline{M}} g(i, n_i^2) - \sum_{i=1}^{\overline{M}} g(i, n_i^1) \\
&= g(\overline{M}, j) + \sum_{i=1}^{\overline{M}-1} g(i, n_i^2) - \sum_{i=1}^{\overline{M}} g(i, n_i^1) \geqslant \\
&= g(\overline{M}, j) - \sum_{i=1}^{\overline{M}} g(i, n_i^1) \\
&= \Delta,
\end{aligned} \tag{5}$$

where the last step follows since $\sum_{i=1}^{\overline{M}-1} g(i, n_i^2) \geqslant 0$. Next, we observe the inequality:

$$g(i, \alpha) \leqslant g(i, \beta), \qquad \text{if} \quad \alpha \leqslant \beta, \qquad i, \alpha \geqslant 1, \tag{6}$$

which is implied by equation (2). Also, by the definitions of $\overline{M}$, n_i, and j, we must have

$$n_i^1 \leqslant j - 1, \qquad \text{for all} \quad i \leqslant \overline{M}, \tag{7}$$

and thus

$$\sum_{i=1}^{\overline{M}} g(i, n_i^1) \leqslant \sum_{i=1}^{\overline{M}} g(i, j - 1). \tag{8}$$

Returning to the evaluation of Δ in equation (5), by the use of (8) we obtain

$$\Delta \geqslant g(\overline{M}, j) - \sum_{i=1}^{\overline{M}} g(i, j - 1) = \Delta', \qquad j \geqslant 2. \tag{9}$$

Now, using (4) and (2), we obtain that $\Delta' = 1$. Thus, by equations (4) and (8), $T(L^2) - T(L^1) \geqslant 1$. Q.E.D.

LEMMA 2. *The transformation T is one to one and onto.*

Proof. The ordering $\succ$ is total and Lemma 1 implies that T is a one to one transformation from $S(M, N)$ into the set of integers I. The least element in $S(M, N)$ is $\underline{L} = (0, 0, 0,..., M)$ and the greatest there is $\bar{L} = (M, 0, 0, 0,..., 0)$. By the definition (4) of T we may verify that $T(\underline{L}) = 0$ while $T(\bar{L}) = \binom{M+N-1}{M} - 1$ (i.e., the smallest and largest elements in $I(M, N)$. Thus the transformation T must be "onto." Q.E.D.

THE INVERSE TRANSFORMATION

The inverse transformation T^{-1} (the existence of which is insured by Lemma 2) can be defined as follows:

Given any number $K \in I(M, N)$, the corresponding composition $L = T^{-1}(K) \in S(M, N)$ is specified uniquely by some sequence of location values $\{n_i\}_{i=1}^{M}$ Accordingly, set $K_M = K$, $K_{i-1} = K_i - \binom{M+N_i-2}{M}$, $i = 2, 3,..., M$, where we construct recursively

$$n_i = \max_n n\colon K_i - \binom{M+n-2}{M} \geqslant 0$$

in the reversed ordering $i = M, M-1, M-2,..., 2, 1$. It is easily verified that $T[T^{-1}(K)] = K$ for all K.

Note. The amount of arithmetical computations in the algorithms for T and T^{-1} can be minimized, in practice, by the usage of the weights provided by the Pascal triangle, as a preset table.

AN APPLICATION

A natural application occurs when it is required to store in a finite memory a function, say f, on the space of N-dimensional probability density. Any probability point is then an N-vector $p = (p_N, p_{N-1},..., p_1)$, where $\sum_{i=1}^{N} p_i = 1$.

It is possible to use a set of compositions $S(M, N)$ to generate a finite grid of points on this space. Probability values are then discretisized by the increment $1/M$ which determines the density of the grid. Each grid point has the form $(1/M)L$, $L \in S(M, N)$, where the Euclidean distance of each such point to any of its nearest neighbors is constant $\sqrt{2}/M$.

The function $f(\cdot)$ is then storable, most compactly, in a one-dimensional array with a minimal number of $\binom{M+N-1}{M}$ memory words. An addressing algorithm that realizes the transformation T (and T^{-1}) provides the correspondence between probability values p and their function values $f(p)$. The transformation, as such, offers also a coding system for probability vectors.

Acknowledgment

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